EFFECT OF VOLUME CONCENTRATION OF INCLUSIONS ON EQUATION OF BUBBLE PULSATIONS IN GAS–LIQUID MEDIA

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In the construction of models of multiphase media the effect of the volume concentration of inclusions on the force of thermal interphase interaction must be taken into account. The theoretical investigation of this problem has been the subject of a great number of studies, which have been reviewed in [1, 2], for instance. It was shown in [1] that interphase interactions, generally speaking, depend on the kind of distribution of inclusions in the medium. The media usually considered are disperse media with two limiting schemes of particular distribution within the medium: with a regular and with a random structure. Disperse media with a regular structure are investigated within the framework of the cell method [1, 2]. Media with a random structure are investigated by using relations connecting the characteristics of interaction of an individual inclusion with the carrier medium for a specific arrangement of the other inclusions and averaging them over the statistical ensemble by using the distribution functions of the inclusions within the medium. At the same time the distribution functions themselves depend on the interaction between the inclusions through the carrier medium. Hence, the problem of determining the distribution functions with due consideration of this interaction is considerably complicated and for its solution, apart from rare exceptions [3], the simplifying assumption of independence of the distribution functions on the interaction of the inclusions is used [4-6]. The problem of determining the mean characteristics of the medium is still fairly complex even when the distribution functions are known and, hence, for its simplification a specific form of interaction of the inclusion with the carrier medium for a fixed distribution of the other particles is assigned approximately. In [3] the description of this interaction was confined to the dipole approximation, while the authors of [5, 6] introduced the substantial assumption of applicability of the mean characteristics of the medium, obtained by averaging over a volume containing many inclusions, for description of the flow near an individual inclusion. In the words of the authors of [5, 6], this assumption is not rigorously justified, but leads to a considerable simplification of the calculations, and in [4] the physically understandable idea of self-consistence was used. The assumptions made in [3-6] do not allow a theoretical estimate of the error that they introduce into the final relations. At the same time the error of the various approximate approaches can be estimated by comparing the final approximate formulas with the accurate formula obtained in these few problems where an accurate result can be obtained. In this paper we consider the problem of vibrations of gas bubbles in an ideal liquid in a case where the volume concentration of bubbles $\alpha_2 \ll 1$. The flow potential of the liquid is written for an arbitrary distribution of the bubbles. The mean values of the potential and the square of the velocity on the surface of the sample bubble for a random distribution of the other particles are calculated to an accuracy of α_2, and a generalized Rayleigh–Lamb equation is obtained. The effect of inaccurate assignment of interaction of the inclusions on the Rayleigh–Lamb equation is analyzed.

We consider a one-velocity monodisperse mixture of a low-viscosity incompressible liquid containing randomly distributed spherical bubbles of radius R on the assumption that the characteristic scale of variation of the mean mixture parameters (R, α_2, etc.) is much greater than R.

The flow potential φ of the liquid for an arbitrary distribution of the centers N of the bubbles contained in volume V_0 bounded by surface Γ_0 is determined as the solution of the following equations with boundary conditions:

$$\Delta\varphi = 0, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma_i} = v_i \quad (i = 0, 1, \ldots, N),$$

where $\Gamma_i (i = 1, \ldots, N)$ is the surface of the i-th bubble; n is the normal to the surface; $v_i (i = 1, \ldots, N)$ is the normal component of the liquid velocity on the boundary of the i-th bubble; v_0 is the normal component of the liquid velocity, due to vibrations of the bubbles, on the boundary of the considered region.

We will find φ by the method of successive approximations of $\varphi = \lim_{n\to\infty} \varphi^n$ by mirror images [7]. We first determine φ^0, the zeroth approximation of φ,

$$\varphi^0 = \sum_{i=1}^{N} \varphi_i^0,$$

where φ_i^0 are determined as the solutions of the following equations with boundary conditions:

$$\Delta\varphi_i^0 = 0, \quad \left.\frac{\partial\varphi_i^0}{\partial n}\right|_{\Gamma_i} = v_i, \quad \varphi_i^0 \to 0 \quad \text{for} \quad |\mathbf{r} - \mathbf{r}_i| \to \infty \quad (i = 1, 2, \ldots, N),$$

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where r is the radius vector of the point at which the value of the potential is determined; r_i is the radius vector of the center of the i-th bubble.

We determine φ^1, the first approximation of φ,

$$\varphi^1 = \varphi^0 + \sum_{i=0}^{N} \varphi_i^1,$$

where φ_i^1 are determined as the solutions of the following equations with boundary conditions:

$$\Delta\varphi_i^1 = 0, \quad \frac{\partial}{\partial n}(\varphi^0 + \varphi_i^1)|_{\Gamma_i} = v_i \quad (i = 0, 1, \ldots, N)$$

for $i \neq 0$, $\varphi_i^1 \to 0$ when $|\mathbf{r} - \mathbf{r}_i| \to \infty$.

The subsequent approximations of φ are obtained in a similar way. It can be shown that the terms φ_i^n decrease as $\varphi_i^n \sim \varphi_i^{n-1}\left(\frac{R}{r_l}\right)^k$, where $k \geqslant 1$; R is the bubble radius; r_l is the distance between the bubbles. Hence, the method of successive approximations gives the potential φ in the form of a series in terms of the parameter R/r_l. We note that the series constructed in this way for the potential around two bubbles converges very rapidly to the exact potential [7-10] for any (including the case of contact) distances between the bubbles.

It follows from the construction that the potential φ is in the form of a sum of terms φ_i, each of which for i = 1, ..., N has no singularities outside the i-th boundary, and φ_0 has no singularities anywhere in the considered region.

The expounded method at each approximation (beginning with the first) takes into account the conditions on the boundary of the mixture. This makes the analysis of the potential difficult. Hence, it is convenient to use $(\varphi_0^1)' = \sum_{n=1}^{\infty} \varphi_0^n$, instead of φ_0^1; the boundary need then be considered only in the first approximation, i.e., $(\varphi_0^n)' = 0$ for $n > 1$. Henceforth we will use the above-described modified algorithm, and will omit the dashes on the corresponding potentials. We will also consider bubbles situated far from the boundary and, hence, for them we can neglect the effect of the specific distribution of the bubbles on (φ_0^1). Then it follows from the construction that φ is in the form of a series whose terms are either independent of the position of any of the bubbles (φ_0^1) or depend on the position of only one (φ_1^0), or on two bubbles (such terms are contained in φ_i^n when $n \geqslant 1$), etc. We sum the terms φ, which depend on the position of precisely l bubbles (l = 0, 1, ..., N), then

$$\varphi(\mathbf{r}\,|\,\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{l=0}^{N} \chi_l^0(\mathbf{r}), \quad \chi_0^0 = \chi_0(\mathbf{r}),$$
$$\chi_l^0 = \sum_{\omega^N_{\mathbf{r}_{i_1},\ldots,\mathbf{r}_{i_l}}} \chi_l(\mathbf{r}\,|\,\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_l}), \tag{1}$$

where χ_l^0 depends on the position of only l bubbles; $\omega^N_{\mathbf{r}_{i_1},\ldots,\mathbf{r}_{i_l}}$ are combinations of l from N bubbles (bubbles $i_1, \ldots, i_l$ are selected); $\chi_l(\mathbf{r}, \mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_l})$, depends only on the position of the bubbles $i_1, \ldots, i_l$.

We note that χ_l^0 was constructed so that it took into account the specificity of interaction of precisely l bubbles, and henceforth we will call χ_l^0 the l-particle interaction.

The bubble dynamics (e.g., the Rayleigh–Lamb equation) is determined by the difference $\langle\varphi_d\rangle = \langle\varphi_b\rangle - \langle\varphi_m\rangle$, where $\langle\varphi_b\rangle$ is the mean liquid potential near the bubble surface; $\langle\varphi_m\rangle$ is the mean liquid potential at a point corresponding to the center of the bubble when it is not there

$$\langle\varphi_b\rangle = \frac{1}{V_0} \iint_{\mathbf{r}_1,\ldots,\mathbf{r}_N} \varphi(\mathbf{r}\,|\,\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_N) f_{N+1}(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_N)\, d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N,$$
$$\langle\varphi_m\rangle = \iint_{\mathbf{r}_1,\ldots,\mathbf{r}_N} \varphi(\mathbf{r}_0\,|\,\mathbf{r}_1, \ldots, \mathbf{r}_N) f_N^m(\mathbf{r}_1, \ldots, \mathbf{r}_N\,|\,\mathbf{r}_0)\, d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N, \tag{2}$$

where $f_k(r_1, \ldots, r_k)$ is the k-particle distribution function; f_N^m is the N-particle distribution function on condition that the point r_0 does not belong to any of the bubbles. We select a system of coordinates with origin at the center of the sample bubble. Function $\langle\varphi_d\rangle$, generally speaking, takes different values at different points on the bubble surface. This leads to a difference in pressure on the bubble surface, which for slow processes (such that the pressure in the bubble is constant) will be balanced by small deformations of the bubble. Hence, to derive the generalized Rayleigh–Lamb equation within the framework of the spherical-bubble model we need to use the quantity $\langle\langle\varphi_d\rangle\rangle$ averaged over the ensemble and the surface of the bubble. For $\langle\langle\varphi_d\rangle\rangle$, using (1) and (2), we can obtain

$$\langle\langle\varphi_d\rangle\rangle = -\frac{R^2\dot{R}}{|\mathbf{r}|} + \sum_{l=1}^{N}\langle\chi_l^b\rangle + \sum_{l=1}^{N}\langle\chi_l^f\rangle, \tag{3}$$

where R is the radius of the bubbles in the mixture and $\dot{R}$ its derivative with respect to time. The first and second terms of Eq. (3) correspond to alteration of the flow due to the presence of the sample bubble at the coordinate origin; the second term corresponds to the regular part of this flow in the vicinity of the sample bubble, while the third term appears owing to the difference in the distribution functions $f_N^m(\mathbf{r}_1, \ldots, \mathbf{r}_N|0)$ and $f_{N+1}(0, \mathbf{r}_1, \ldots, \mathbf{r}_N)$

$$\langle\chi_l^b\rangle = (\alpha_2)^l\left(\frac{3}{4\pi R^3}\right)^l \frac{1}{V_0}\iint\limits_{\mathbf{r}_1,\ldots,\mathbf{r}_l} \chi_{l+1}^b(0|\mathbf{r}_1, \ldots, \mathbf{r}_l)\, f_{l+1}(0, \mathbf{r}_1, \ldots, \mathbf{r}_l)\, d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_l, \tag{4}$$

$$\langle\chi_l^f\rangle = (\alpha_2)^l\left(\frac{3}{4\pi R^3}\right)^l \iint\limits_{\mathbf{r}_1,\ldots,\mathbf{r}_l} \chi_l(0|\mathbf{r}_1, \ldots, \mathbf{r}_l)\left[f_l^m(\mathbf{r}_1, \ldots, \mathbf{r}_l|0) - f_{l+1}(0, \mathbf{r}_1, \ldots, \mathbf{r}_N)\right] d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_l,$$

where $\chi_{l+1}^b(0|\mathbf{r}_1, \ldots, \mathbf{r}_l)$ are the terms in the potential [due to $l+1$ particle interaction of the (1) sample bubble, and l other bubbles] having singularities in any bubble, except the sample bubble.

It is apparent from (4) that to determine $\langle\langle\varphi_d\rangle\rangle$, accurate to $(\alpha_2)^1$ it is sufficient to know the two-particle (binary) distribution function and the two-particle interaction and for determination of $\langle\langle\varphi_d\rangle\rangle$, accurate to $(\alpha_2)^2$, we need to know the three-particle distribution function and the three-particle interaction, etc.

Representation (4) has sense if the integrals $\langle\chi_l^b\rangle$and$\langle\chi_l^f\rangle$ have a finite limit when $\alpha_2 \to 0$ and $V_0 \to \infty$. We note that for a disperse medium of regular structure the distribution functions f^m and f have sharp spikes and these integrals tend to infinity when $\alpha_2 \to 0$. Whence it follows that expansion (4) of $\langle\langle\varphi_d\rangle\rangle$in terms of $(\alpha_2)^l$ is not valid for disperse media with a regular structure (it can be shown that the first terms of the correct expansion of $\langle\chi_1^l\rangle$ and $\langle\chi_1^f\rangle$in this case are proportional to $\alpha_2^{1/3}$).

We will show that for media with a random structure expansion (4) is valid. To investigate the convergence of the coefficients of $(\alpha_2)^1$ the behavior of the integrands when $|\mathbf{r}_1| \to \infty$ is important and, hence, in (4) we can confine ourselves to the main terms of the expansion χ_{1+1}^bin terms of $R/|\mathbf{r}_1|$

$$\chi_1 = -\frac{R^2(\mathbf{r}_1)\dot{R}(\mathbf{r}_1)}{|\mathbf{r}_1|}, \quad \chi_{1+1}^b \sim R\dot{R}\left(\frac{R}{|\mathbf{r}_1|}\right)^4.$$

It is apparent from (4) that for convergence of the integrals $\langle\chi_1^b\rangle$and$\langle\chi_1^f\rangle$ it is sufficient that the binary correlative function $f_2(0, \mathbf{r}_1)$ is bounded $[f_2(0, \mathbf{r}_1) < k_1]$ and differs from the one-particle distribution function $f_1^m(\mathbf{r}_1|0)$ by not more than $k_2/|\mathbf{r}_1|^3$, where k_1 and k_2 are constants. We note that the binary correlative function, corresponding to the rigid-sphere model [11], has exponential convergence to the one-particle function, and the binary correlative function obtained from the interaction of inclusions with the carrier medium in the Stokes approximation [3] also ensures convergence of the coefficients of α_2 in (4).

Let the l-particle distribution function $f_l(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ satisfy the following conditions:

$$f_{l+1}(0, \mathbf{r}_1, \ldots, \mathbf{r}_l) < k_3, \quad |f_{l+1}(0, \mathbf{r}_1, \ldots, \mathbf{r}_l) - f_l^m(\mathbf{r}_1, \ldots, \mathbf{r}_l|0)| < \frac{k_4}{[\min(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l)]^3},$$

where k_3 and k_4 are constants. In the investigation of the convergence of the integrals for $(\alpha_2)^2$ in (4) we can confine ourselves, as for the integrals considered above, to the main terms in $R/|\mathbf{r}_1|$ in the expansion of χ_2 and χ_{2+1}^b, which have the form

$$\chi_2 \sim R\dot{R}\left(\frac{R}{|\mathbf{r}_2 - \mathbf{r}_1|}\right)^2\left(\frac{R}{|\mathbf{r}_1|}\right)^2\cos\theta, \quad \chi_{2+1}^b \sim R\dot{R}\left(\frac{R}{|\mathbf{r}_1|}\right)^2\left(\frac{R}{|\mathbf{r}_2 - \mathbf{r}_1|}\right)^3\left(\frac{R}{|\mathbf{r}_2|}\right)^2, \tag{5}$$

where θ is the angle between the vector $\mathbf{r}_1$ and the vector $\mathbf{r}_2 - \mathbf{r}_1$. Using (5) for a point not lying at the boundary of the region V_0 we can show the convergence of the coefficients of α_2^2. Estimates from the modulus of χ_2 (as, for instance, χ_{2+1}^b is estimated) are insufficient for convergence of the integral $\langle\chi_2^f\rangle$. In a similar way it can be shown that the coefficients of $(\alpha_2)^2$ converge. The above arguments show that in expansion (4) the decrease in the terms when $\alpha_2 \to 0$ is correctly shown.

We derive a generalized Rayleigh–Lamb equation accurate to α_2. In this case we can neglect the potential φ_0^1 contained in $\chi_{l+1}^b(0|\mathbf{r}_1, \ldots, \mathbf{r}_l)$, $\chi_l(0|\mathbf{r}_1, \ldots, \mathbf{r}_l)$, which is proportional to α_2 and, hence, as (4) shows, gives a contribution at least quadratic in α_2 to the Rayleigh–Lamb equation.

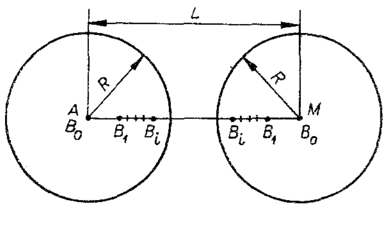

Fig. 1

The analysis conducted above has shown that to determine $\langle\langle\varphi_d\rangle\rangle$ accurate to $(\alpha_2)^1$ it is sufficient to know only the two-particle interaction. Hence we find the liquid flow potential φ_2 for two growing bubbles of radius R whose centers are at distance L. Using the method of successive approximations by mirror images [7, 8] we find that φ_2 can be represented as a sum of potentials of point sources and distributed sinks lying in a straight line connecting the bubble centers. The point sources are at points B_i (see Fig. 1) at distances L_i from the bubble centers. The quantities L_i and the source powers C_i are given by the following recurrent relations: $L_i = R^2/(L - L_{i-1})$, $L_0 = 0$; $C_i = C_{i-1}L_i/R$, $C_0 = R^2\dot{R}$. The sinks are uniformly distributed on the segments B_iB_{i+1} with a power per unit length $d_i = C_{i+1}/(L_{i+1} - L_i)$. The described potential is an exact solution of the posed problem. It allows determination of the two-particle interaction of the sample bubble A with bubble M (see Fig. 1), situated at distance L. As (3) shows, the interaction χ^b_{1+1} agrees with the potential due to the whole system of point sources and distributed sinks situated in the bubble M, except the source at the center of bubble M, which corresponds to one-particle interaction.

In the calculation of $\langle\langle\varphi_d\rangle\rangle$ accurate to α_2 we can neglect the corrections linear in α_2 to the binary correlative function and use the following simple expression [11], obtained for the case of non-interacting spheres:

$$f_2(\mathbf{r}_1) = \begin{cases} 0, & R < |\mathbf{r}_1| < 2R, \\ 1, & 2R < |\mathbf{r}_1|. \end{cases} \tag{6}$$

When the effect of bubble interaction on the binary correlative function is proportional to α_2, Eq. (6) can also be used to derive the Rayleigh–Lamb equation.

Integrals $\langle\chi_1^b\rangle$ and $\langle\chi_1^f\rangle$ can be calculated numerically, accurate to α_2. The final formula for $\langle\langle\varphi_d\rangle\rangle$ between the mean potential on the bubble surface and the mean potential in the liquid has the form

$$\langle\langle\varphi_d\rangle\rangle \approx - R^2\dot{R}/|\mathbf{r}| + 3.6\alpha_2 R\dot{R}. \tag{7}$$

We substitute the value of $\langle\langle\varphi_d\rangle\rangle$ in the Cauchy–Lagrange integral

$$\frac{\partial}{\partial t}\langle\langle\varphi_d\rangle\rangle\Big|_{|\mathbf{r}|=R} + \frac{\dot{R}^2}{2} + \frac{\langle v_\tau^2\rangle}{2} + \frac{p_2 - 2\sigma/R}{\rho} = \frac{\langle v^2\rangle}{2} + \frac{p_1}{\rho}, \tag{8}$$

where $\langle v_\tau^2\rangle$ is the mean square of the tangential velocity component of the liquid on the bubble surface; $\langle v^2\rangle$ is the mean square of the liquid velocity at a point corresponding to the center of the sample bubble; p_2 and p_1 are the pressure in the bubble and liquid; σ, ρ are, respectively, the coefficient of surface tension and density of the liquid. In the deviation of (8) we used the fact that the normal component of the liquid velocity on the bubble surface, on the basis of assumption of its sphericity, is equal to $\dot{R}$.

From (1) we can obtain the following expressions for $\langle v^2\rangle$ and $\langle v_\tau^2\rangle$:

$$\langle v^2\rangle = \int_{\mathbf{r}_1,\ldots,\mathbf{r}_N}\!\!\int \left[\sum_{l=0}^{\infty} \nabla\chi_l\right]^2 f_N^m(\mathbf{r}_1, \ldots, \mathbf{r}_N \,|\, 0)\, d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N;$$

$$\langle v_\tau^2\rangle = \frac{1}{V_0}\int_{\mathbf{r}_1,\ldots,\mathbf{r}_N}\!\!\int\int_S\!\!\int \left(\sum_{l=0}^{\infty} \nabla\chi_{l+1}^b\right)^2 f_{N+1}(0 \,|\, \mathbf{r}_1, \ldots, \mathbf{r}_N)\, dS d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N,$$

where S is the surface of the sample bubble.

Making estimates similar to those made in the deduction of Eq. (4) we can show that $\langle v^2\rangle$ and $\langle v_\tau^2\rangle$ can be expanded in a series in terms of α_2. The values of $\langle v^2\rangle$ and $\langle v_\tau^2\rangle$ are determined, accurate to $(\alpha_2)^1$, only by the one-particle and two-particle interaction, respectively. Then

$$\langle v^2\rangle = \frac{3}{R^3}\int_{L=R}^{\infty}\left(\frac{R^2\dot{R}}{L^2}\right)^2 L^2 dL,\quad \langle v_\tau^2\rangle = \frac{3}{R^3}\int_{L=2R}^{\infty}\int_S v_\tau^2 dS dL,$$

where v_τ is the tangential component of the liquid velocity on the bubble surface, determined by the flow potential φ_2, when the distance between the bubbles is L.

The value of $\langle v^2\rangle$ is determined analytically, and $\langle v_\tau^2\rangle$ is calculated numerically:

$$\langle v^2\rangle \approx 3\alpha_2\dot{R}^2,\quad \langle v_\tau^2\rangle \approx 3.25\alpha_2\dot{R}^2. \tag{9}$$

Substituting (7), (9) in (8), we obtain the generalized Rayleigh–Lamb equation

$$R\ddot{R}(1-3.6\alpha_2)+\frac{3}{2}(\dot{R})^2(1-9.7\alpha_2)=\frac{p_2-2\sigma/R-p_1}{\rho}. \tag{10}$$

The conducted analysis and the accurately derived Rayleigh–Lamb equation (10) enable us to analyze the error introduced by the simplifications often used in the theory of disperse media with randomly distributed inclusions.

It is apparent from (4) that corrections quadratic in α_2 to the generalized Rayleigh–Lamb equation depend, generally speaking, on the three-particle interaction of the inclusions, the three-particle distribution function and the conditions on the region boundary (φ_0^1).

We analyze the error introduced by inaccurate assignment of the two-particle interaction between the inclusions. If we use (as in [3]) only the main terms in R/r_l in calculation of the interaction between the bubbles, the generalized Rayleigh–Lamb equation will take the form

$$R\ddot{R}(1-4.5\alpha_2)+\frac{3}{2}(\dot{R})^2(1-12\alpha_2)=\frac{p_2-2\sigma/R-p_1}{\rho}. \tag{11}$$

The generalized Rayleigh–Lamb equation has the same form when the idea of self-consistency, adopted in [4], is used. A comparison of (10) and (11) shows that the error in determination of the coefficients is approximately 20%. In view of the considerable simplifications obtained by the use of these assumptions [Eq. (11) is derived analytically], however, they can be recommended for an approximate assessment of the effect of volume concentration on the interphase interaction.

The idea of self-consistency and some other assumptions enable us to obtain a correction, quadratic in α_2, to the Rayleigh–Lamb equation. As was shown above, this correction cannot be determined accurately, which raises doubts as to the error of the linear term. In view of the great physical clarity of the self-consistent method and the improved agreement with experiment obtained in some studies [4], however, we can expect that when these corrections are taken into account these methods will allow an approximate calculation of the coefficients of $(\alpha_2)^2$, as was the case with corrections linear in α_2.

LITERATURE CITED

1. R. I. Nigmatulin, Fundamentals of Mechanics of Heterogeneous Media [in Russian], Nauka, Moscow (1978).
2. J. Happel and H. Brenner, Low Reynolds Number Hydrodynamics, Noordhoff International Publ., Leiden (1973).
3. G. K. Batchelor and J. T. Green, "The determination of bulk stress in a suspension of spherical particles to order c^2," J. Fluid Mech., 56, Part 3, 401 (1972).
4. A. M. Golovin and V. E. Chizhov, "Calculation of rate of precipitation of a homogeneous suspension," Prikl. Mat. Mekh., No. 1 (1978).
5. Yu. A. Buevich and V. G. Markov, "Continual mechanics of monodisperse suspensions. Rheological equations of state for suspensions of measured concentration," Prikl. Mat. Mekh., No. 6 (1973).
6. Yu. A. Buevich and Yu. A. Korneeva, "Heat and mass transfer in a disperse medium," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1974).
7. H. Lamb, Hydrodynamics, University Press, Cambridge (1932).
8. L. M. Milne-Thomson, Theoretical Hydrodynamics, MacMillan, New York (1960).
9. O. V. Voinov, "Movement of two spheres of variable radii in an ideal fluid," in: Scientific Conference [in Russian], Inst. Mekh. Mosk. Gos. Univ., Moscow (1970).
10. R. M. Garipov, "Closed equations of motion of a liquid with bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1973).
11. A. M. Golovin and V. E. Zhizhov, "Calculation of binary correlative function in a two-phase system," Prikl. Mat. Mekh., No. 6 (1977).